

# Multi-tier Network Performance Analysis using a Shotgun Cellular System

Prasanna Madhusudhanan\*, Juan G. Restrepo†, Youjian (Eugene) Liu\*, Timothy X Brown\*+, and Kenneth R. Baker+

\* Department of Electrical, Computer and Energy Engineering, † Department of Applied Mathematics, + Interdisciplinary Telecommunications Program

University of Colorado, Boulder, CO 80309-0425 USA

{mprasanna, juanga, eugeneliu, timxb, ken.baker}@colorado.edu

**Abstract**—This paper studies the carrier-to-interference ratio ( $\frac{C}{I}$ ) and carrier-to-interference-plus-noise ratio ( $\frac{C}{I+N}$ ) performance at the mobile station (MS) within a multi-tier network composed of  $M$  tiers of wireless networks, with each tier modeled as the *homogeneous*  $n$ -dimensional ( $n$ -D,  $n=1,2$ , and 3) shotgun cellular system, where the base station (BS) distribution is given by the homogeneous Poisson point process in  $n$ -D. The  $\frac{C}{I}$  and  $\frac{C}{I+N}$  at the MS in a single tier network are thoroughly analyzed to simplify the analysis of the multi-tier network. For the multi-tier network with given system parameters, the following are the main results of this paper: (1) semi-analytical expressions for the tail probabilities of  $\frac{C}{I}$  and  $\frac{C}{I+N}$ ; (2) a closed form expression for the tail probability of  $\frac{C}{I}$  in the range  $[1, \infty)$ ; (3) a closed form expression for the tail probability of an approximation to  $\frac{C}{I}$  in the entire range  $[0, \infty)$ ; (4) a lookup table based approach for obtaining the tail probability of  $\frac{C}{I+N}$ , and (5) the study of the effect of shadow fading and BSs with ideal sectorized antennas on the  $\frac{C}{I}$  and  $\frac{C}{I+N}$ . Based on these results, it is shown that, in a practical cellular system, the installation of additional wireless networks (microcells, picocells and femtocells) with low power BSs over the already existing macrocell network will always improve the  $\frac{C}{I+N}$  performance at the MS.

**Index Terms**—Multi-tier networks, Cellular Radio, Co-channel Interference, Fading channels, Poisson point process.

## I. INTRODUCTION

The modern cellular communication network is a complex overlay of heterogeneous networks such as macrocells, microcells, picocells, femtocells, etc. The base station (BS) distribution appears increasingly irregular as the density of BSs grows over time while bounded by cell site limitation. Due to computational constraints, system designers cannot study the overall network at once, and have to resort to simulations for specific portions of the network. As it is hard to obtain insight and general conclusions from such studies, it is desirable to abstract and simplify the model. At one end of the abstraction, the BSs are assumed to be at the centers of regular hexagonal cells. At the other end, the BS deployments are modeled according to a Poisson point process. In [1], the author makes a connection between the ideal hexagonal cellular system and the cellular system with the BS placement according to a homogeneous Poisson point process on a plane (two dimensions, 2-D), called the shotgun cellular system (SCS). It is shown that the carrier-to-interference ratio, ( $\frac{C}{I}$ ), of the SCS lower bounds that of the ideal hexagonal cellular system and moreover, they converge in the strong shadow fading regime.

We have explored the SCS in detail in [1]–[3]. The utility of the SCS model in the study of the cognitive radio networks can be found in [4].

In this paper, we study the practical cellular system by viewing the macrocells, microcells, picocells and femtocells as the different tiers of a multi-tier network. We focus on the  $\frac{C}{I}$  and the carrier-to-interference-plus-noise ratio ( $\frac{C}{I+N}$ ) at the mobile station (MS) in a multi-tier network with  $M$  tiers of heterogeneous networks (hence called an  $M$ -tier network). The BS distribution of the practical cellular system follows regular topologies (e.g. to match the customer density patterns along highways, between suburbs and city centers and within large multi-storey buildings). Each tier of the  $M$ -tier network is modeled as the *homogeneous*  $l$ -dimensional ( $l$ -D,  $l=1,2$ , and 3) SCS, where the BS distribution is according to the homogeneous Poisson point process in  $\mathbb{R}^l$ ,  $l=1,2,3$ . In the *homogeneous*  $l$ -D SCS,  $l=1$  is a model for the highway scenario,  $l=2$  models the planar deployment of BSs in suburbs, and  $l=3$  models the BS deployments within large multi-storey buildings and wireless LANs (WLAN) in multi-storey residential areas. A Poisson point process in  $\mathbb{R}^2$  has been a popular model adopted in the literature for the locations of nodes in the study of ad hoc and other uncoordinated networks ([5]–[7] are a few selected references). It has also been used in studying two-tier networks composed of macrocells and femtocells [8], [9]. Here, we characterize the cellular performance in a multi-tier network with BS distributions according to the Poisson point process in  $\mathbb{R}^1$ ,  $\mathbb{R}^2$  and  $\mathbb{R}^3$ . In [10], the authors study the multi-tier network with the BS distribution in the various tiers according to the Poisson point process in  $\mathbb{R}^2$ , and derive a closed form expression for the tail probability of  $\frac{C}{I}$  in the range  $[1, \infty)$  for the special case of Rayleigh fading. In this paper, we characterize the  $\frac{C}{I}$  in the entire range  $[0, \infty)$  and for any general fading distribution.

**Contributions of the paper:** Firstly, we emphasize that the study of the cellular performance of the multi-tier network is tightly coupled with a similar study on a single tier network. Hence, we indulge in thoroughly understanding the single tier network and its properties. Sections III, IV and V deal with the single tier network. In Section VI, based on the theory developed in the previous sections, we completely characterize the signal quality at the MS in a  $M$ -tier network measured in terms of the carrier-to-interference ratio ( $\frac{C}{I}$ ) and the carrier-

to-interference-plus-noise ratio  $\left(\frac{C}{I+N}\right)$ . In particular, for the multi-tier network, we derive (1) semi-analytical expressions for the tail probabilities of  $\frac{C}{I}$  and  $\frac{C}{I+N}$ ; (2) a closed form expression for the tail probability of  $\frac{C}{I}$  in the range  $[1, \infty)$ ; (3) a closed form expression for the tail probability of an approximation to  $\frac{C}{I}$  in the entire range  $[0, \infty)$ ; (4) a lookup table based approach for obtaining the tail probability of  $\frac{C}{I+N}$ , and (5) the effect of shadow fading and BSs with ideal sectorized antennas on the  $\frac{C}{I}$  and  $\frac{C}{I+N}$ . Finally, it is shown that the installation of additional wireless networks (microcells, picocells and femtocells) with low power BSs over the already existing macrocell network will always improve the  $\frac{C}{I+N}$  performance at the MS.

## II. SYSTEM MODEL

This section describes the various elements used to model the wireless network, namely, the BS layout, the radio environment, and the performance metrics of interest.

**BS Layout:** We define the SCS and *homogeneous l-D SCS* ( $l=1,2,3$ ) and describe the model for the single tier and multi-tier networks.

**Definition 1.** The *Shotgun Cellular System (SCS)* is a model for the cellular system in which the BSs are placed in a given  $l$ -dimensional plane ( $l = 1, 2$ , and  $3$ ) according to a Poisson point process on  $\mathbb{R}^l$ . The intensity function of the Poisson point process is called the BS density function in the context of the SCS. (See [3] for more details.)

**Definition 2.** In the *homogeneous l-D SCS* ( $l \in \{1, 2, 3\}$ ), the BSs are placed according to a homogeneous Poisson point process on  $\mathbb{R}^l$  with a BS density  $\lambda_0$ , such that the probability that there exists a BS in a small region  $\mathcal{H} \subseteq \mathbb{R}^l$  is  $\lambda_0 |\mathcal{H}|$ , where  $|\mathcal{H}| \ll 1$  is the length, area or volume of the region  $\mathcal{H}$  for  $l = 1, 2$ , and  $3$ , respectively; and the events in non-overlapping regions are independent of each other.

**Radio Environment:** The signal from the BS undergoes path-loss and shadow fading; and is also affected by background noise. The signal power at a distance  $R$  from the BS is given by  $P = K\Psi R^{-\varepsilon}$ , where  $K$  captures the transmission power and the antenna gain of the BS,  $\Psi$  is the random shadow fading factor, and  $R^{-\varepsilon}$  represents the inverse power law path-loss with  $\varepsilon$  as the path-loss exponent, and  $R$  as the distance from the BS. The noise power in the system is  $N$ .

**Single tier network:** In this paper, the single-tier network refers to the macrocell network and the BS layout is according to the *homogeneous l-D SCS*,  $l = 1, 2, 3$ .

**Multi-tier ( $M$ -tier) network:** The  $M$ -tier network is assumed to be composed of  $M$  independent *homogeneous l-D SCSs* with BS density  $\{\lambda_i\}_{i=1}^M$ , for each tier. For the  $M$ -tier network,  $K$  and the cumulative density function (c.d.f.) of  $\Psi$  are different for each tier.

**Performance Metric:** In this paper, we are concerned with the signal quality at a MS within the wireless network. The MS is assumed to be located at the origin of  $\mathbb{R}^l$ ,  $l = 1, 2, 3$  in which the multi-tier network is defined. The MS receives signals from all the BSs, and chooses to communicate with the BS that corresponds to the strongest received signal power. This

BS is referred to as the “serving BS”, and all the other BSs are collectively called the “interfering BSs”. Consequently, the signal quality at the MS is defined as the ratio of the received power from the serving BS (denoted by  $C$  or  $P_S$ ) to the sum of the total interference power (denoted by  $I$  or  $P_I$ , sum of the powers from the interfering BSs) and the noise power ( $N$ ), and is called the carrier-to-interference-plus-noise ratio  $\left(\frac{C}{I+N}\right)$ . In an “interference limited system”,  $I \gg N$  and the signal quality is referred to as the carrier-to-interference ratio  $\left(\frac{C}{I}\right)$ . Thus, for a single tier network, the  $\frac{C}{I}$  and  $\frac{C}{I+N}$  are

$$\frac{C}{I} \stackrel{(a)}{=} \frac{K_S \Psi_S R_S^{-\varepsilon}}{\sum_{i=1}^{\infty} K_i \Psi_i R_i^{-\varepsilon}}, \quad \frac{C}{I+N} \stackrel{(b)}{=} \frac{K_S \Psi_S R_S^{-\varepsilon}}{\sum_{i=1}^{\infty} K_i \Psi_i R_i^{-\varepsilon} + N}, \quad (1)$$

where subscript “ $S$ ” denotes the serving BS and subscript “ $i$ ” indexes the interfering BSs;  $K_S = \{K_i\}_{i=1}^{\infty}$  are the transmission powers that can be equal a constant or independent and identically distributed (i.i.d.) random variables;  $R_S$  and  $\{R_i\}_{i=1}^{\infty}$  are random variables that come from the underlying Poisson point process that governs the BS placement;  $\Psi_S$  and  $\{\Psi_i\}_{i=1}^{\infty}$  are i.i.d. random variables. Hence,  $\frac{C}{I}$  and  $\frac{C}{I+N}$  are random variables, and can be characterized by a probability density function (p.d.f.), c.d.f. or the tail probability. The tail probability of  $\frac{C}{I+N}$  is given by  $\mathbb{P}\left(\left\{\frac{C}{I+N} > \eta\right\}\right)$ , and is the probability that a MS in the SCS has a signal quality of at least  $\eta$ ,  $\eta \geq 0$ . In the following section, we characterize the tail probability of the  $\frac{C}{I}$  at the MS in a single tier network.

## III. $\frac{C}{I}$ CHARACTERIZATION FOR A SINGLE TIER NETWORK

Here, the transmission power and the antenna gains of all the BSs in the SCS are assumed to be constant (say,  $K$ ). Also, the shadow fading factors are assumed to be unity. Hence, from the expression for  $\frac{C}{I}$  in (1a), the BS closest to the MS is the serving BS and the expression for  $\frac{C}{I}$  is

$$\frac{C}{I} = \frac{K R_1^{-\varepsilon}}{\sum_{i=2}^{\infty} K R_i^{-\varepsilon}}, \quad (2)$$

where  $R_1 \leq R_2 \leq R_3 \dots$  are the distances between the BSs and the MS, arranged in a non-decreasing order. Further, recall that the BS layout in the single tier network is as in the *homogeneous l-D SCS* ( $l = 1, 2, 3$ ) with BS density  $\lambda_0$ . Thus, the p.d.f. of  $R_1$  is given by  $f_{R_1}(r_1) = \lambda_0 b_l r_1^{l-1} e^{-\frac{\lambda_0 b_l r_1^l}{l}}$ , where  $r_1 \geq 0$  and  $b_l = 2, 2\pi, 4\pi$  for  $l = 1, 2, 3$ , respectively, and the conditional p.d.f. of the  $i^{\text{th}}$  closest BS conditioned on the  $(i-1)^{\text{th}}$  closest BS, is  $f_{R_i|R_{i-1}}(r_i|r_{i-1}) = \lambda_0 b_l r_i^{l-1} e^{-\frac{\lambda_0 b_l (r_i^l - r_{i-1}^l)}{l}}$ ,  $r_i \geq r_{i-1}$ .

**Theorem 1.** In a homogeneous  $l$ -D SCS with a constant BS density  $\lambda_0$ , if the path-loss exponent satisfies  $\varepsilon > l$ ,

(a) the characteristic function of  $P_I$  conditioned on  $R_1$  is

$$\Phi_{P_I|R_1}(\omega|r_1) = \exp\left(\frac{\lambda_0 b_l r_1^l}{l} \left(1 - {}_1F_1\left(-\frac{l}{\varepsilon}; 1 - \frac{l}{\varepsilon}; \frac{i\omega K}{r_1^\varepsilon}\right)\right)\right), \quad (3)$$

(b) the characteristic function of  $(\frac{C}{T})^{-1}$  is given by

$$\begin{aligned}\Phi_{(\frac{C}{T})^{-1}}(\omega) &= E_{R_1} \left[ \Phi_{P_I|R_1} \left( \frac{\omega}{P_S} \middle| R_1 \right) \right] \\ &= \frac{1}{{}_1F_1 \left( -\frac{l}{\varepsilon}; 1 - \frac{l}{\varepsilon}; i\omega \right)},\end{aligned}\quad (4)$$

where  $E_{R_1}$  is the expectation w.r.t.  $R_1$ , and  ${}_1F_1(\cdot; \cdot; \cdot)$  is called the confluent hypergeometric function of the first kind.

*Proof:* See [3, Corollary 2]. ■

The significance of Theorem 1 is in the following remarks.

**Remark 1.** The tail probability of  $\frac{C}{T}$  may be directly obtained from the characteristic function and is given by

$$\begin{aligned}\mathbb{P} \left( \left\{ \frac{C}{T} > \eta \right\} \right) &= \begin{cases} \int_{\omega=-\infty}^{\infty} \Phi_{(\frac{C}{T})^{-1}}(\omega) \left( \frac{1 - \exp(-\frac{i\omega}{\eta})}{i\omega} \right) \frac{d\omega}{2\pi}, & \eta > 0 \\ 1, & \eta = 0. \end{cases}\end{aligned}\quad (5)$$

*Proof:* See [2, Eq. (9)]. ■

**Remark 2.** The characteristic function of the  $(\frac{C}{T})^{-1}$  does not depend on  $\lambda_0$ , and hence the tail probability of  $\frac{C}{T}$  at a MS in the *homogeneous l-D SCS* does not depend on  $\lambda_0$ .

**Remark 3.** The characteristic function of  $(\frac{C}{T})^{-1}$  for a *homogeneous 2-D* and *3-D SCS* is the same as that of a *homogeneous 1-D SCS* with path-loss exponents  $\frac{\varepsilon}{2}$  and  $\frac{\varepsilon}{3}$ , respectively. Hence, the corresponding  $\frac{C}{T}$  performances are identical.

Remark 2 proves why the curves corresponding to the *homogeneous 1-D, 2-D and 3-D SCSs* in Fig. 1 are straight lines. Remark 3 helps build an intuition of why the *homogeneous 1-D SCS* has a higher tail probability of  $\frac{C}{T}$  than *homogeneous 2-D and 3-D SCSs*; Fig. 1 now corresponds to comparing the tail probabilities of  $\frac{C}{T}$  in a *homogeneous 1-D SCS* with path-loss exponents  $\varepsilon$ ,  $\frac{\varepsilon}{2}$ , and  $\frac{\varepsilon}{3}$ , respectively. As the path-loss exponent decreases, the BSs farther away from the MS have a greater contribution to the total interference power at the MS, and this leads to a poorer  $\frac{C}{T}$  at the MS and a smaller tail probability (computed by evaluating the integral in (5)). An important consequence of Remark 3 is as follows.

**Corollary 1.** For a homogeneous *l-D SCS*,  $l = 1, 2, 3$ , where the path-loss exponent is  $\varepsilon$ , the tail probability of  $\frac{C}{T}$  is

$$\mathbb{P} \left( \left\{ \frac{C}{T} > \eta \right\} \right) = \mathbb{P} \left( \left\{ \frac{C}{T} > 1 \right\} \right) \times \eta^{-\frac{l}{\varepsilon}} \quad (6)$$

$$= \mathcal{K}_{\frac{\varepsilon}{l}} \eta^{-\frac{l}{\varepsilon}}, \quad \forall \eta \geq 1, \varepsilon > l, \quad (7)$$

where  $\mathcal{K}_{\frac{\varepsilon}{l}}$  is a constant parametrized by  $\frac{\varepsilon}{l}$ .

*Proof:* In [1], we have shown that

$$\mathbb{P} \left( \left\{ \frac{C}{T} > \eta \right\} \right) = \mathbb{P} \left( \left\{ \frac{C}{T} > 1 \right\} \right) \times \eta^{-\frac{l}{\varepsilon}}, \quad (8)$$

where  $\eta \geq 1$ , and  $\varepsilon > 2$ , for a *homogeneous 2-D SCS*. From Remark 3, (8) holds for all *homogeneous 1-D SCS* with path-loss exponent  $\frac{\varepsilon}{2}$  and therefore, for all *homogeneous 3-D SCSs* with path-loss exponent  $\frac{3\varepsilon}{2}$ . Hence, (6) hold true, and (7) is obtained by noting that the characteristic function of  $(\frac{C}{T})^{-1}$  is a function of  $\frac{\varepsilon}{l}$  and so  $\mathbb{P} \left( \left\{ \frac{C}{T} > 1 \right\} \right)$  is a constant. ■

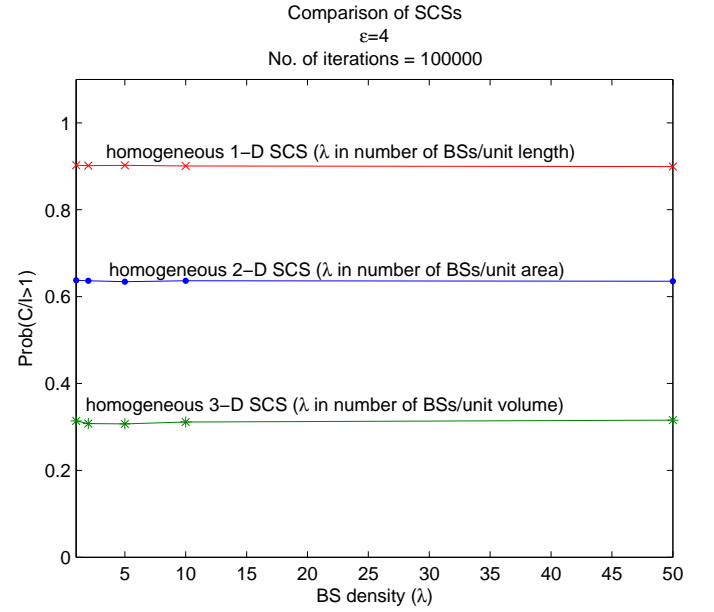


Figure 1. Invariance of  $\frac{C}{T}$  of the *homogeneous l-D SCS* w.r.t. BS density ( $\lambda$ ). The  $\frac{C}{T}$  tail probability is independent of  $\lambda$  (proved in Remark 2), and the *homogeneous 1-D SCS* has a better  $\frac{C}{T}$  tail probability than the *homogeneous 2-D and 3-D SCS* (proved in Remark 3).

The constant  $\mathcal{K}_{\frac{\varepsilon}{l}}$  can be obtained by using (4) and evaluating the integral in (5) with  $\eta = 1$ . Note that  $\frac{C}{T}$  is a non-negative random variable with a support of  $[0, \infty)$ , and surprisingly, its tail probability has such a simple form as given by (7) in the region  $[1, \infty)$ . Next, we define the so-called *few BS approximation* and derive closed form expressions for the tail probability of  $\frac{C}{T}$  at MS in a *homogeneous l-D SCS* for both the regions  $[0, 1)$  and  $[1, \infty)$ .

**Definition 3.** The *few BS approximation* corresponds to modeling the total interference power at the MS in the SCS as the sum of the contributions from the strongest few interfering BSs and an ensemble average of the contributions of the rest of the interfering BSs.

Recall from (2) that  $P_I = \sum_{i=2}^{\infty} K R_i^{-\varepsilon}$ , where  $\{R_i\}_{i=1}^{\infty}$  is the set of distances of BSs arranged in the ascending order of their separation from the MS. The arrangement also corresponds to the descending order of their contribution to  $P_I$ . In the *few BS approximation*,  $P_I$  is approximated by  $\tilde{P}_I(k) = \sum_{i=2}^k K R_i^{-\varepsilon} + E \left[ \sum_{i=k+1}^{\infty} K R_i^{-\varepsilon} \middle| R_k \right]$ , for some  $k$ , where  $E[\cdot]$  is the expectation operator and refers to the ensemble average of the contributions of BSs beyond  $R_k$ . The  $\frac{C}{T}$  at the MS obtained by the *few BS approximation* is denoted by  $\frac{C}{I_k}$ . Next, we study  $\frac{C}{I_k}$  for the *homogeneous l-D SCSs*.

**Lemma 1.** For the *homogeneous l-D SCS*, with BS density  $\lambda_0$  and  $\varepsilon > l$ , for  $k=1, 2, 3$ ,

$$E \left[ \sum_{i=k+1}^{\infty} K R_i^{-\varepsilon} \middle| R_k \right] = \frac{\lambda_0 b_l K R_k^{l-\varepsilon}}{\varepsilon - l}. \quad (9)$$

*Proof:* See [3, Corollary 4]. ■

Next, the tail probability of  $\frac{C}{I_2} = \frac{C}{I_k} \bigg|_{k=2}$  is derived.

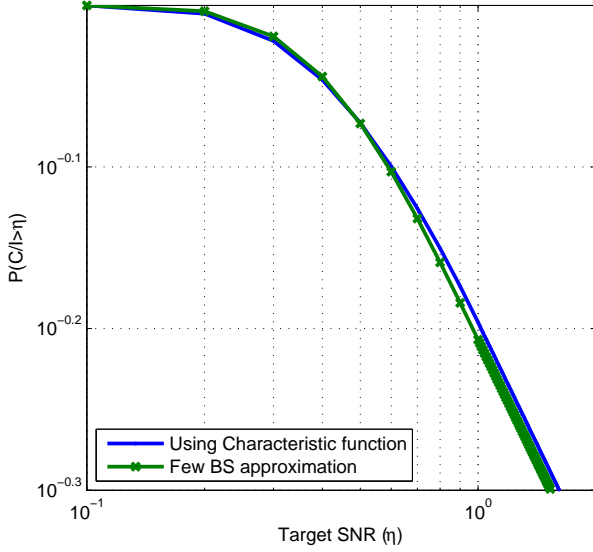


Figure 2. Homogeneous 2-D SCS: Comparing  $\frac{C}{T}$  and  $\frac{C}{I_2}$ ,  $\varepsilon = 4$

**Theorem 2.** In the homogeneous  $l$ -D SCS with BS density  $\lambda_0$  and path-loss exponent  $\varepsilon$  ( $\varepsilon > l$ ), the tail probability of  $\frac{C}{I_2}$  is

$$\mathbb{P}\left(\left\{\frac{C}{I_2} > \eta\right\}\right) \quad (10)$$

$$= \begin{cases} \eta^{-\frac{1}{\varepsilon}} C_{\frac{\varepsilon}{\varepsilon-1}}, & \eta \geq 1 \\ 1 - \frac{(1+u(\eta))}{e^{u(\eta)}} + \eta^{-\frac{1}{\varepsilon}} D_{\frac{\varepsilon}{\varepsilon-1}}(\eta), & \eta < 1 \end{cases}, \quad (11)$$

where  $u(\eta) = (\frac{\varepsilon}{\varepsilon-1} - 1) \left(\frac{1}{\eta} - 1\right)$ ,  $C_{\frac{\varepsilon}{\varepsilon-1}} = G(0, \infty)$ ,  $D_{\frac{\varepsilon}{\varepsilon-1}}(\eta) = G(u(\eta), \infty)$ , and  $G(a, b) = \int_{v=a}^b \frac{ve^{-v}}{(1+v(\frac{\varepsilon}{\varepsilon-1}-1))^{-1}} \frac{1}{\varepsilon} dv$ .

*Proof:* See [3, Theorem 2]. ■

Notice that  $\mathbb{P}\left(\left\{\frac{C}{T} > \eta\right\}\right) = \frac{K_{\frac{\varepsilon}{\varepsilon-1}}}{C_{\frac{\varepsilon}{\varepsilon-1}}} \mathbb{P}\left(\left\{\frac{C}{I_2} > \eta\right\}\right)$  for  $\eta \geq 1$ .

Fig. 2 shows the comparison of the tail probabilities of  $\frac{C}{T}$  (computed using the characteristic function of  $(\frac{C}{T})^{-1}$ ) and  $\frac{C}{I_2}$  for the homogeneous 2-D SCS with path-loss exponent 4. Notice that the gap between the two tail probability curves is negligible in the region  $\eta \in [0, 1]$ , and further, both the curves are straight lines parallel to each other in the region  $\eta \in [1, \infty)$ , when the tail probability is plotted against  $\eta$ , both in the logarithmic scale. This shows that  $\frac{C}{I_2}$  is a good approximation for  $\frac{C}{T}$  and can be characterized in closed form.

#### IV. $\frac{C}{I+N}$ IN A SINGLE TIER NETWORK

Here, as in Section III, the transmission powers of all BSs are constant and shadow fading factors are equal to unity. We first obtain the tail probability of  $\frac{C}{I+N}$  using the characteristic function of  $(\frac{C}{I+N})^{-1}$  derived in the following corollary.

**Corollary 2.** In a homogeneous  $l$ -D SCS with BS density  $\lambda_0$  and path-loss exponent  $\varepsilon$  ( $\varepsilon > l$ ), the characteristic function of the sum of the total interference power ( $P_I$ ) and noise power ( $N$ ) conditioned on  $R_1$  is  $\Phi_{P_I+N|R_1}(\omega|r_1) = \exp(i\omega N) \times$

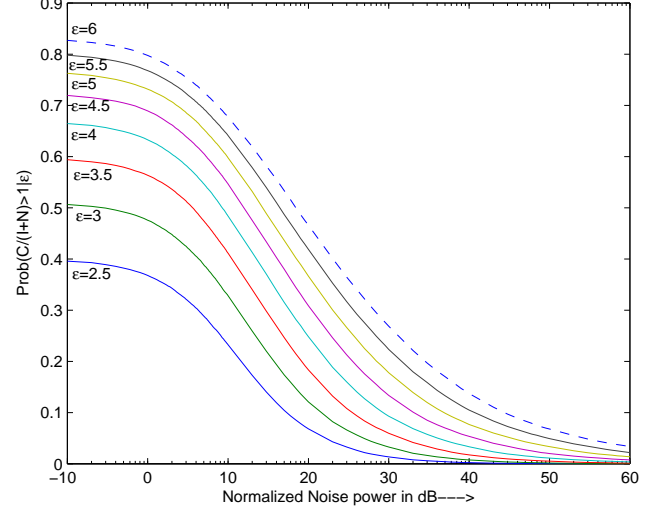


Figure 3.  $\mathbb{P}\left(\left\{\frac{C}{I+N} > 1\right\}\right)$  vs Normalized noise power  $(N\lambda_0^{-\frac{\varepsilon}{\varepsilon-1}} K^{-1})$  for  $l = 2$

$\Phi_{P_I|R_1}(\omega|r_1)$ , and the characteristic function of  $(\frac{C}{I+N})^{-1}$  is  $\Phi_{(\frac{C}{I+N})^{-1}}(\omega) = E_{R_1} \left[ \exp\left(i\frac{\omega}{P_S} N\right) \times \Phi_{P_I|R_1}\left(\frac{\omega}{P_S} | R_1\right) \right]$ , where  $\Phi_{P_I|R_1}(\omega|r_1)$  is given by (3).

*Proof:* The expressions for  $\Phi_{P_I+N|R_1}(\omega|r_1)$  and  $\Phi_{(\frac{C}{I+N})^{-1}}(\omega)$  follow directly from the definition of characteristic function, where  $N$  is a constant. ■

Further, the tail probability of  $\frac{C}{I+N}$  is obtained by substituting  $\frac{C}{T}$  with  $\frac{C}{I+N}$  in (5). Next, an interesting property of the  $\frac{C}{I+N}$  at the MS in the homogeneous  $l$ -D SCS is presented.

**Corollary 3.** If the  $\frac{C}{I+N}$  at the MS in the homogeneous  $l$ -D SCS is specified by  $(\lambda_0, \varepsilon, K, N)$  where  $\lambda_0$  is the BS density,  $\varepsilon$  is the path-loss exponent,  $K$  is the constant transmission power of each BS, and  $N$  is the constant noise power, then,

$$\frac{C}{I+N} \Big|_{(\lambda_0, \varepsilon, K, N)} \stackrel{=st}{=} \frac{C}{I+N} \Big|_{(1, \varepsilon, 1, N')}, \quad (12)$$

where  $N' = N / (\lambda_0^{\frac{\varepsilon}{\varepsilon-1}} K)$  and “=st” means that the c.d.f’s are same.

*Proof:* See [3, Corollary 9]. ■

So, it is sufficient to analyze the homogeneous  $l$ -D with  $\lambda_0 = K = 1$  and maintain a table for the tail probability of  $\frac{C}{I+N}$  for different values of  $N'$  and  $\varepsilon$ . We can find the  $\frac{C}{I+N}$  at the MS for a homogeneous  $l$ -D SCS with any given  $(\lambda_0, \varepsilon, K, N)$  by just reading out the tail probability of  $\frac{C}{I+N}$  corresponding to  $\varepsilon$  and  $N'$  obtained using Corollary 3 from the lookup table. The lookup table is presented for a homogeneous 2-D SCS in Fig. 3 as a plot of  $\mathbb{P}\left(\left\{\frac{C}{I+N} > 1\right\}\right)$  against  $N'$  for different values of  $\varepsilon$ . Further, in the homogeneous  $l$ -D SCS, as  $\lambda_0$  increases, the noise power  $N'$  of the equivalent SCS decreases according to Corollary 3, and in the limit as  $\lambda_0 \rightarrow \infty$ ,  $N'$  approaches zero and hence  $\frac{C}{I+N} \xrightarrow{D} \frac{C}{I}$ , where  $\xrightarrow{D}$  corresponds to convergence in distribution. Thus, in an

“interference limited system” (large  $\lambda_0$ ), the signal quality is measured in terms of  $\frac{C}{T}$ . Next, we study the effect of shadow fading on the  $\frac{C}{T}$  and  $\frac{C}{T+N}$  at the MS in a single tier network.

### V. SHADOW FADING

Theorem 3 analytically shows that the effect of the introduction of shadow fading to the SCS is completely captured in the BS density of the *homogeneous l-D SCS*.

**Theorem 3.** When shadow fading in the form of i.i.d non-negative random factors,  $\{\Psi_i\}$ , is introduced to the homogeneous *l-D SCS* with BS density  $\lambda_0$ , for the  $\frac{C}{T}$  and  $\frac{C}{T+N}$  analysis, the resulting system is equivalent to another homogeneous *l-D SCS* with BS density  $\lambda_0 E[\Psi^{\frac{1}{\epsilon}}]$ , as long as  $E[\Psi^{\frac{1}{\epsilon}}] < \infty$ .

*Proof:* The expression for  $\frac{C}{T}$  and  $\frac{C}{T+N}$  in (1a) and (1b) may be written as  $\frac{C}{T} = \frac{\bar{R}_1^{-\epsilon}}{\sum_{k=2}^{\infty} \bar{R}_k^{-\epsilon}}$  and  $\frac{C}{T+N} = \frac{\bar{R}_1^{-\epsilon}}{\sum_{k=2}^{\infty} \bar{R}_k^{-\epsilon} + N}$ , where  $\bar{R}_1 = R_S \Psi_S^{-\frac{1}{\epsilon}}$  and  $\bar{R}_{k+1} = R_k \Psi_k^{-\frac{1}{\epsilon}}$ ,  $k = 1, 2, 3 \dots$ . Now, the expression for  $\frac{C}{T}$  is similar to the no shadow fading case in (2), with the  $\bar{R}$ 's replacing the  $R$ 's. Using the Marking theorem of Poisson process in [11, Page 55],  $\bar{R} = R \Psi^{-\frac{1}{\epsilon}}$  follows the homogeneous Poisson process in  $\mathbb{R}^l$  with intensity  $\lambda_0 E[\Psi^{\frac{1}{\epsilon}}]$ . For a complete proof, see [3, Theorem 4]. ■

Further, the  $\frac{C}{T}$  and  $\frac{C}{T+N}$  at the MS in the *homogeneous l-D SCS* with shadow fading is the same as that in the equivalent *homogeneous l-D SCS* where there is no shadow fading. The following remark illustrates the consequence of the theorem on the  $\frac{C}{T}$  and  $\frac{C}{T+N}$  at the MS.

**Remark 4.** In the *homogeneous l-D SCS* with BS density  $\lambda_0$ ,

- (a) shadow fading has no effect on the  $\frac{C}{T}$  at the MS, and,
- (b) the effect of shadow fading is completely captured in the noise power term of the  $\frac{C}{T+N}$ .

*Proof:* Firstly, using Theorem 3, it is sufficient to analyze the  $\frac{C}{T}$  and  $\frac{C}{T+N}$  for the *homogeneous l-D SCS* with BS density  $\lambda_0 E[\Psi^{\frac{1}{\epsilon}}]$ . Then, Remark 4(a) follows from Remark 2. Finally, since the  $\frac{C}{T+N}$  in this case has the same c.d.f. as the equivalent *homogeneous l-D SCS* in Corollary 3 with  $N' = N K^{-1} \left( \lambda_0 E[\Psi^{\frac{1}{\epsilon}}] \right)^{-\frac{1}{\epsilon}}$ , Remark 4(b) is proved. ■

**Example 1.** Consider a *homogeneous 2-D SCS* with an average BS density  $\lambda_0$ , where each BS is affected by an i.i.d log-normal shadow fading factor with a mean 0 and standard deviation  $\sigma$ . Using Theorem 3, the equivalent *homogeneous 2-D SCS* has a BS density  $\bar{\lambda}_0 = \lambda_0 \exp\left(\frac{2\sigma^2}{\epsilon^2}\right)$ . Note that  $\bar{\lambda}_0 \geq \lambda_0$ ,  $\forall \sigma, \epsilon$ , and from Remark 4, the introduction of shadow fading improves the  $\frac{C}{T+N}$  performance at the MS measured in terms of the tail probability of  $\frac{C}{T}$ .

Next, we study the  $\frac{C}{T}$  and the  $\frac{C}{T+N}$  at the MS in a multi-tier network based on the analysis for the single tier network modelled as the *homogeneous l-D SCS*.

### VI. MULTI-TIER NETWORKS (M-TIER NETWORKS)

All the BSs of the  $i^{\text{th}}$  tier of a  $M$ -tier network are assumed to have constant transmission power,  $\{\kappa_i\}_{i=1}^M$ . Firstly, the  $M$ -tier network is reduced to an equivalent single tier network.

**Theorem 4.** Consider a multi-tier network consisting of  $M$  independent homogeneous *l-D SCS* with BS density  $\{\lambda_i\}_{i=1}^M$ , such that all the BSs in  $i^{\text{th}}$  tier have a constant transmission power  $\kappa_i$ , then, this multi-tier network is equivalent to a single tier network (homogeneous *l-D SCS*) with

- (a) BS density  $\lambda_0 = \sum_{i=1}^M \lambda_i$ , and,
- (b) the transmission power of each BS is an i.i.d. random variable  $K = \kappa_i$  with a probability  $p_i = \frac{\lambda_i}{\lambda_0}$ ,  $i = 1, 2, \dots, M$ .

*Proof:* Theorem 4 (a) follows directly from the Superposition theorem of Poisson process in [11, Page 16], and holds true even as  $M \rightarrow \infty$ . Next, consider a region  $\mathcal{H} \subseteq \mathbb{R}^l$ , and let  $\mathbb{P}_i(\mathcal{H})$  denotes the probability of finding one BS belonging to the  $i^{\text{th}}$  tier conditioned on the event that there exists one BS in  $\mathcal{H}$ . Then,  $\mathbb{P}_i(\mathcal{H}) = \frac{\mathbb{P}(\{N_i=1\})}{\mathbb{P}(\{N_0=1\})} \times \prod_{j=1, j \neq i}^M \mathbb{P}(\{N_j=0\})$ , where  $\{N_i\}_{i=0}^M$  is the set of random variables denoting the number of BSs in  $\mathcal{H}$  for the *homogeneous l-D SCS* with BS density  $\{\lambda_i\}_{i=0}^M$  ( $\lambda_0$  is defined in Theorem 4(a)), respectively. Note that  $N_i \sim \text{Poisson}(\lambda_i |\mathcal{H}|)$ , where  $|\mathcal{H}|$  is the length, area or volume of  $\mathcal{H}$  for  $l = 1, 2$ , and  $3$ , respectively. Further,  $\mathbb{P}(\mathcal{H}) = \frac{\lambda_0}{\lambda_0}$  is independent of  $\mathcal{H}$  and hence,  $\mathbb{P}(\{K = \kappa_i | 1 \text{ BS in } \mathcal{H}\}) = \frac{\lambda_i}{\lambda_0}$ . ■

The following remarks result due to Theorem 4.

**Remark 5.** The equivalent *homogeneous l-D SCS* in Theorem 4, with BS density  $\lambda_0$  and i.i.d. random transmission powers can further be reduced to the *homogeneous l-D SCS* with BS density  $\lambda_0 E[K^{\frac{1}{\epsilon}}] = \lambda_0 \sum_{i=1}^M p_i \kappa_i^{\frac{1}{\epsilon}}$  and unity transmission powers at all BSs.

*Proof: (Outline)* In (1a) and (1b),  $\Psi_S$  and  $\{\Psi_i\}_{i=1}^{\infty}$  are equal to unity;  $K_S$  and  $\{K_i\}_{i=1}^{\infty}$  are i.i.d. discrete random variables with the probability mass function (p.m.f.) of  $K$  (Theorem 4 (b)). Now, follow the same steps in the proof of Theorem 3 with  $\bar{R} = R K^{-\frac{1}{\epsilon}}$  to obtain the result. ■

**Remark 6.** For the multi-tier network, Theorem 1 and Remark 1 together give the tail probability of  $\frac{C}{T}$ , and Remark 2 shows that the  $\frac{C}{T}$  is independent of the  $\lambda_0$ ,  $\{p_i\}_{i=1}^M$ , and  $\{\kappa_i\}_{i=1}^M$ . Further, Corollary 1 gives the closed form expression for the tail probability of  $\frac{C}{T}$  in  $[1, \infty)$ , and Theorem 2 gives the closed form expression for  $\frac{C}{T}$  under few BS approximation.

Since the *homogeneous l-D SCS* (homogeneous Poisson point process in  $\mathbb{R}^l$ ) has the maximum entropy for a given mean number of points in any subset of  $\mathbb{R}^l$ , Remark 6 shows that even the most arbitrary placement of BSs in  $\mathbb{R}^l$  does not degrade the  $\frac{C}{T}$  performance, and hence any intelligent strategy in BS placement in any of the tiers of the multi-tier network will only improve the  $\frac{C}{T}$ .

**Remark 7.** The  $\frac{C}{T+N}$  of the multi-tier network has the same c.d.f. as that of the equivalent *homogeneous l-D SCS* in Corollary 3 with  $N' = N \left( \lambda_0 E[K^{\frac{1}{\epsilon}}] \right)^{-\frac{1}{\epsilon}}$ . Further, the tail probability of  $\frac{C}{T+N}$  can be computed using Corollary 2 and (5) (by replacing  $\frac{C}{T}$  with  $\frac{C}{T+N}$ ).

Note that  $\frac{C}{T}$  and  $\frac{C}{T+N}$  studied in this section, for a multi-tier network with  $M = 2$  corresponds to a 2-tier network with macrocell network and femtocell network with all femtocell BSs operating in the open access mode. Further, an important consequence of Remark 7 is as follows.

**Corollary 4.** Inclusion of additional tiers of wireless networks with low transmission power BSs over an existing wireless network will only improve the  $\frac{C}{I+N}$  performance of the overall network. Further, as the BS density of the additional tiers increases,  $\frac{C}{I+N}$  performance keeps improving, and approaches the  $\frac{C}{I}$  performance as the BS density approaches infinity.

*Proof:* The existing wireless network is a single tier network with BS density  $\lambda_1$  with a constant transmission power  $\kappa_1$  at all the BSs. Hence, the  $\frac{C}{I+N}$  has the c.d.f. as the equivalent homogeneous  $l$ -D SCS in Corollary 3 with  $N' = N_1 = N\lambda_1^{-\frac{1}{l}}\kappa_1^{-1}$ . Now, let  $M-1$  additional wireless networks be installed on top of this single tier network to form an  $M$ -tier network, with BS densities  $\{\lambda_i\}_{i=2}^{M-1}$  and constant transmission powers  $\{\kappa_i\}_{i=2}^M$ . From Remark 7, this  $M$ -tier network has the same c.d.f. as the equivalent homogeneous  $l$ -D SCS in Corollary 3 with  $N' = N_2 = N_1 \left(1 + \sum_{i=2}^M \frac{\lambda_i}{\lambda_1} \cdot \left(\frac{\kappa_i}{\kappa_1}\right)^{\frac{1}{l}}\right)^{-\frac{1}{l}} \leq N_1$ . Thus, the  $M$ -tier network has a smaller noise power, which leads to an improved  $\frac{C}{I+N}$  performance of the overall network compared to the existing wireless network (c.f. Fig. 3). Further, as  $\{\lambda_i\}_{i=2}^M$  increases,  $N_2$  decreases, and converges to zero as at least one of  $\{\lambda_i\}_{i=2}^M$  approaches  $\infty$ . Then, the  $\frac{C}{I+N}$  converges, in distribution, to the  $\frac{C}{I}$  of a single tier network (Section III). ■

In a practical cellular system, the macrocell BSs have large transmission powers in order to provide cellular coverage, and the microcell, picocell and femtocell BSs have relatively smaller transmission powers. In this case, Corollary 4 applies; hence the installation of these networks with low power BSs will not harm the existing cellular performance and any intelligent strategy will only improve it.

*Remark 8.* When i.i.d. shadow fading factors  $\{\Psi_i\}$  independent of the BS placement random process are introduced to the multi-tier network,  $\frac{C}{I}$  is the same as in a multi-tier network without shadow fading and  $\frac{C}{I+N}$  has the same c.d.f. as that of the equivalent homogeneous  $l$ -D SCS in Corollary 3 with  $N' = N \left(\lambda_0 E \left[K^{\frac{1}{l}}\right] \times E \left[\Psi^{\frac{1}{l}}\right]\right)^{-\frac{1}{l}}$ , as long as  $E \left[\Psi^{\frac{1}{l}}\right] < \infty$ , where  $E[\cdot]$  is the expectation operator.

*Proof: (Outline)* The multi-tier network is equivalent to a single tier network with BS density  $\lambda_0$  (Theorem 4(a)). Next, in (1a) and (1b),  $K_S$  and  $\{K_i\}_{i=1}^{\infty}$  are i.i.d. discrete random variables with the p.m.f. of  $K$  (Theorem 4 (b));  $\Psi_S$  and  $\{\Psi_i\}_{i=1}^{\infty}$  are i.i.d. random factors with the p.d.f. of  $\Psi$ . Now, follow the same steps in the proof of Theorem 3 with  $\bar{R} = RK^{-\frac{1}{l}}\Psi^{-\frac{1}{l}}$ , and reduce the equivalent single tier network to another single tier network with BS density  $\lambda_0 E \left[K^{\frac{1}{l}}\right] E \left[\Psi^{\frac{1}{l}}\right]$ , unity transmission powers at all BSs and no shadow fading. Finally, use Remark 6 and Remark 7 to complete the proof. ■

**Corollary 5.** (Ideal sectorized antennas) If each BS in the  $i^{\text{th}}$  tier of the  $M$ -tier network has BSs with ideal sectorized antennas with an antenna gain,  $G_i$ , and beam-width  $\theta_i$ , such that the BSs antenna faces the MS with probability  $\frac{\theta_i}{2\pi}$ , in which case the transmission power is  $K_i = G_i \times X_i$ , where  $X_i \sim \text{Bernoulli}(\frac{\theta_i}{2\pi})$  for each  $i = 1, 2, \dots, M$ ,

the equivalent homogeneous  $l$ -D SCS will have BSs with transmission powers which are i.i.d. random variables and have a probability mass function (p.m.f.):  $\mathbb{P}(\{K = G_i\}) = \frac{p_i \theta_i}{2\pi}$ ,  $i = 1, 2, \dots, N$ , and  $\mathbb{P}(\{K = 0\}) = 1 - \sum_{i=1}^M \frac{p_i \theta_i}{2\pi}$ .

Further, the  $\frac{C}{I}$  and  $\frac{C}{I+N}$  of this multi-tier network can be computed using Remark 6 and Remark 7. Finally, Corollary 4 and Remark 8 also hold true for this multi-tier network.

## VII. CONCLUSIONS

In this paper, we study the  $\frac{C}{I}$  and  $\frac{C}{I+N}$  at the MS within a multi-tier network, where each tier is modeled as the *homogeneous*  $l$ -D SCS ( $l = 1, 2$ , and  $3$ ). Most studies of wireless networks model the network as the homogeneous Poisson point process in  $\mathbb{R}^2$ . Here, we study the wireless network with the BS distribution according to the homogeneous Poisson point process in  $\mathbb{R}^1$  and  $\mathbb{R}^3$  as well, and highlight their significance in practical scenarios.

The  $\frac{C}{I}$  and  $\frac{C}{I+N}$  in a single tier network are thoroughly analyzed. Using these results, we completely characterize the  $\frac{C}{I}$  and the  $\frac{C}{I+N}$  at the MS within a multi-tier ( $M$ -tier) network. This paper brings together and refines a set of results on *homogeneous*  $l$ -D SCS to demonstrate how the SCS model (developed in [1]–[3]) can easily handle the case of multi-tier networks. The main takeaway from this paper is due to Corollary 4: in a practical cellular system, installation of additional wireless networks (microcells, picocells and femtocells) with low power BSs over the already existing macrocell network will always improve the signal quality at the MS, measured in terms of the tail probability of  $\frac{C}{I+N}$ .

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